North-Holland

NOTE

IDENTITIES CONTAINING GAUSSIAN BINOMIAL COEFFICIENTS

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1. Introduction

The Gaussian binomial coefficient $\left[{n \atop r}\right]$ is defined (see [1, p. 218]) by

$$\left[{n \atop r}\right]=\begin{cases}\dfrac{q^n-1}{q-1}\dfrac{q^{n-1}-1}{q^2-1}\cdots\dfrac{q^{n+1-r}-1}{q^r-1}, & \text{if } 0<r\leqslant n,\\ 1, & \text{if } r=0,\\ 0, & \text{if } r<0 \text{ or } r>n,\end{cases}$$

where q is a variable. From this definition, it is easy to deduce the following properties

(i) $\lim\limits_{q\to 1}\left[{n \atop r}\right]=\binom{n}{r}$, binomial coefficient.

(ii) $\left[{n \atop r}\right]=\left[{n \atop n-r}\right]$.

The purpose of this note is to establish an identity containing Gaussian binomial coefficients. From this result, we shall deduce a generalization of a combinatorial identity obtained in [2].

Throughout the paper, the function $A(X_1, X_2, \ldots, X_n)$ is defined by

$$A(X_1, X_2, \ldots, X_n)=\sum\frac{n!}{(k_1)!\cdots(k_n)!}\left(\frac{X_1}{1}\right)^{k_1}\cdots\left(\frac{X_n}{n}\right)^{k_n},$$

where the sum is overall nonnegative integral values of k_1 to k_n such that $k_1+2k_2+\cdots+nk_n=n$.

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2. Preliminary results

First of all, we define the function $[t]_n$ by

$$[t]_n = \begin{cases} t(t-1)\left(t-\dfrac{q^2-1}{q-1}\right)\cdots\left(t-\dfrac{q^{n-1}-1}{q-1}\right), & \text{if } n \geqslant 1, \\ 1, & \text{if } n = 0, \end{cases} \tag{1}$$

and let

$$t^n = \sum_{r=0}^{n} S_q(n, r)[t]_r, \quad n \geqslant 1. \tag{2}$$

From these notions, we have

$$\lim_{q \to 1} [t]_r = t(t-1)\cdots(t-r+1) = (t)_r,$$

the factorial function and

$$\lim_{q \to 1} S_q(n, r) = S(n, r),$$

the Stirling numbers of the second kind. Now from (2), we obtain $S_q(n, m) = 1$ and $S_q(n, 0) = 0$ for all n. Also $S_q(n, r) = 0$ for $r > n$ and $r < 0$. Moreover, $S_q(n, r)$ obeys the recurrence relation

$$S_q(n+1, r) = S_q(n, r-1) + \frac{q^r - 1}{q-1} S_q(n, r). \tag{3}$$

Now we are in position to prove two results which are used in the next section.

Lemma 1. *Let k and n be any positive integers. Then*

$$S_q(k+n, k+1) + \sum_{i=2}^{n} (-1)^{i-1}\left(\frac{q^{k+1}-1}{q-1}\right)\cdots\left(\frac{q^{k+i-1}-1}{q-1}\right)S_q(k+n, k+i)$$
$$= S_q(k+n-1, k).$$

Proof. Using (3), the left member of the above equation is equal to

$$S_q(k+n, k+1) + \sum_{i=2}^{n} (-1)^{i-1}\left(\frac{q^{k+1}-1}{q-1}\right)\cdots\left(\frac{q^{k+i-1}-1}{q-1}\right)$$
$$\times\left\{S_q(k+n-1, k+i-1) + \left(\frac{q^{k+i}-1}{q-1}\right)S_q(k+n-1, k+i)\right\}$$
$$= S_q(k+n, k+1) + \sum_{i=2}^{n} (-1)^{i-1}\left(\frac{q^{k+1}-1}{q-1}\right)\cdots\left(\frac{q^{k+i-1}-1}{q-1}\right)$$
$$\times S_q(k+n-1, k+i-1)$$
$$+ \sum_{i=3}^{n+1} (-1)^{i}\left(\frac{q^{k+1}-1}{q-1}\right)\cdots\left(\frac{q^{k+i-1}-1}{q-1}\right)S_q(k+n-1, k+i-1)$$

$$= S_q(k+n, k+1) - \left(\frac{q^{k+1}-1}{q-1}\right) S_q(k+n-1, k+1)$$
$$= S_q(k+n-1, k).$$

Lemma 2. *Let $r \geq 0$ be an integer and let k, n be any positive intergers such that $n > k + r$. Let $f(x)$ be any polynomial of degree n. Then the coefficient of x in the expression*

$$\frac{f(x)}{x^r(x-1)\left(x-\frac{q^2-1}{q-1}\right)\cdots\left(\frac{q^k-1}{q-1}\right)}$$

is given by

$$\sum_{j=1}^{n-k-r} \frac{f^{(k+j+r)}(0)}{(k+j+r)!} S_q(k+j-1, k).$$

Proof. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\frac{f(x)}{x^r(x-1)\cdots\left(x-\frac{q^k-1}{q-1}\right)}$$
$$= a_n \frac{x^{n-r}}{(x-1)\cdots\left(x-\frac{q^k-1}{q-1}\right)} + a_{k+r+1}\frac{x^{k+1}}{(x-1)\cdots\left(x-\frac{q^k-1}{q-1}\right)}$$
$$+ \cdots + \frac{a_0}{(x-1)\cdots\left(x-\frac{q^k-1}{q-1}\right)x^r}.$$

Consequently to prove our result, it suffices to find the coefficient of x in the expression

$$\frac{x^{k+j}}{(x-1)\left(x-\frac{q^2-1}{q-1}\right)\cdots\left(x-\frac{q^k-1}{q-1}\right)}, \quad j = 1, 2, \ldots, n-k-r. \tag{4}$$

Using the identity

$$x^{k+j} = \sum_{i=0}^{k+j} S_q(k+j, i)[x]_i,$$

we get

$$\frac{x^{k+j}}{(x-1)\cdots\left(x-\frac{q^k-1}{q-1}\right)} = \sum_{i=0}^{k+j} \frac{S_q(k+j, i)[x]_i}{(x-1)\cdots\left(x-\frac{q^k-1}{q-1}\right)}$$

and thus the coefficient of x in (4) is

$$S_q(k+j,k+1)+\sum_{i=2}^{j}(-1)^{i-1}\left(\frac{q^{k+1}-1}{q-1}\right)\cdots\left(\frac{q^{k+i-1}-1}{q-1}\right)S_q(k+j,k+i),$$

which is equal to $S_q(k+j-1,k)$ by Lemma 1. □

3. Principal theorem

Theorem 1. *Let n, r by any positive integers. Let f be any polynomial of degree m. Then*

$$\sum_{i=1}^{n}\frac{(-1)^{i+1}\begin{bmatrix} n\\ i\end{bmatrix}q^{\frac{1}{2}i(i-1)}f\left(\frac{q^i-1}{q-1}\right)}{q^{i(n-1)}\left(\frac{q^i-1}{q-1}\right)^r}$$

$$=\frac{1}{r!}\left[f^{(r)}(0)+\sum_{j=1}^{r}\binom{r}{j}f^{(r-j)}(0)A(T_1,\ldots,T_j)\right]$$

$$+(-1)^{n+1}\frac{(q^n-1)\cdots(q-1)}{(q-1)^n}\sum_{j=1}^{m-n-r+1}\frac{f^{(n+j+r-1)}(0)}{(n+j+r-1)!}S_q(n+j-1,n),$$

where $T_m=\sum_{j=1}^{n}\left(\frac{q-1}{q^j-1}\right)^m$, $1\leqslant m\leqslant r$, and in the case $m-n-r<0$, the empty sum is to be interpreted as meaning 0.

Proof. Since $f(x)$ is a polynomial of degree m, then we have

$$\frac{f(x)}{x^r(x-1)\cdots\left(x-\frac{q^n-1}{q-1}\right)}$$

$$=b_mx^{m-n-r}+\cdots+b_{n+r}+\frac{h(x)}{x^r(x-1)\cdots\left(x-\frac{q^n-1}{q-1}\right)}, \tag{5}$$

where the degree of $h(x)$ is $<n+r$. But from the partial expansion

$$\frac{(-1)^n(q^n-1)\cdots(q-1)h(x)}{x^r(x-1)\cdots\left(x-\frac{q^n-1}{q-1}\right)}$$

$$=\frac{C_1}{x}+\frac{C_2}{x^2}+\cdots+\frac{C_r}{x^r}+\frac{D_1}{x-1}+\cdots+\frac{D_n}{\left(x-\frac{q^n-1}{q-1}\right)},$$

we can write, for $1 \leqslant i \leqslant n$,

$$D_i = \lim_{x \to (q^i - 1/q - 1)} \left\{ \frac{(-1)^n (q^n - 1) \cdots (q-1) h(x) \left(x - \dfrac{q^i - 1}{q - 1}\right)}{x^r (x-1) \cdots \left(x - \dfrac{q^i - 1}{q-1}\right) \cdots \left(x - \dfrac{q^n - 1}{q - 1}\right)} \right\}$$

$$= \frac{(-1) \begin{bmatrix} n \\ i \end{bmatrix} q^{\frac{1}{2}i(i-1)} (q-1)^n h\left(\dfrac{q^i - 1}{q-1}\right)}{\left(\dfrac{q^i - 1}{q - 1}\right)^{r-1} q^{i(n-1)}}.$$

Then we have

$$\sum_{i=1}^{n} \frac{(-1)^i \begin{bmatrix} n \\ i \end{bmatrix} q^{\frac{1}{2}i(i-1)} (q-1)^n h\left(\dfrac{q^i - 1}{q - 1}\right)}{\left(\dfrac{q^i - 1}{q-1}\right)^{r-1} q^{i(n-1)} \left(x - \dfrac{q^i - 1}{q - 1}\right)}$$

$$= \frac{(-1)^n (q^n - 1) \cdots (q - 1) h(x)}{x^r (x-1) \cdots \left(x - \dfrac{q^n - 1}{q-1}\right)} - \sum_{j=1}^{r} \frac{C_j}{x^j}. \tag{6}$$

However from (5), we have $f\left(\dfrac{q^i - 1}{q - 1}\right) = h\left(\dfrac{q^i - 1}{q-1}\right)$. Substituting this fact in (6) and using this new equation, Eq. (5), becomes, when x tends to 0,

$$\sum_{i=1}^{n} \frac{(-1)^{i+1} \begin{bmatrix} n \\ i \end{bmatrix} q^{\frac{1}{2}i(i-1)} (q-1)^n f\left(\dfrac{q^j - 1}{q-1}\right)}{q^{i(n-1)} \left(\dfrac{q^i - 1}{q-1}\right)^r}$$

$$= (-1)^{n+1} (q^n - 1) \cdots (q-1) b_{n+r}$$

$$+ \lim_{x \to 0} \left\{ \frac{(-1)^n (q^n - 1) \cdots (q-1) f(x)}{x^r (x-1) \cdots \left(x - \dfrac{q^n - 1}{q-1}\right)} - \sum_{j=1}^{r} \frac{C_j}{x^j} \right\} \tag{7}$$

To evaluate this limit we use the expansion

$$\frac{(-1)^n (q^n - 1) \cdots (q-1) f(x)}{x^{r+1} (x-1) \cdots \left(x - \dfrac{q^n - 1}{q-1}\right)}$$

$$= d_m x^{m-n-r-1} + \cdots + d_{n+r+1} + \frac{A_1}{x} + \cdots + \frac{A_{r+1}}{x^{r+1}} + R(x), \tag{8}$$

where $R(x)$ denotes the sum of the fractions corresponding to $(x-1) \cdots$

$\left(x-\dfrac{q^n-1}{q-1}\right)$. It is easy to prove that $A_{j+1}=C_j$, for $j=1,2,\ldots,r$. Therefore from (8) we get

$$\begin{aligned}A_1&=\lim_{x\to 0}\left\{\frac{(-1)^n(q^n-1)\cdots(q-1)f(x)}{x^r(x-1)\cdots\left(x-\dfrac{q^n-1}{q-1}\right)}-\sum_{j=1}^{r}\frac{A_{j+1}}{x^j}\right\}\\&=\lim_{x\to 0}\left\{\frac{(-1)^n(q^n-1)\cdots(q-1)f(x)}{x^r(x-1)\cdots\left(x-\dfrac{q^n-1}{q-1}\right)}-\sum_{j=1}^{r}\frac{C_j}{x^j}\right\}\\&=\frac{1}{r!}\lim_{x\to 0}\frac{d^r}{dx^r}\left\{\frac{(-1)^n(q^n-1)\cdots(q-1)f(x)}{(x-1)\cdots\left(x-\dfrac{q^n-1}{q-1}\right)}\right\}.\end{aligned}$$

Now from Leibniz's rule, we obtain

$$\begin{aligned}A_1=&\frac{(-1)^n(q^n-1)\cdots(q-1)}{r!}\sum_{j=0}^{r}\binom{r}{j}f^{(r-j)}(0)\\&\times\left\{\lim_{x\to 0}\left((x-1)^{-1}\cdots\left(x-\frac{q^n-1}{q-1}\right)^{-1}\right)^{(j)}\right\},\end{aligned}\tag{9}$$

and using the derivative of composite functions [3], we have

$$A_1=\frac{(q-1)^n}{r!}\left\{f^{(r)}(0)+\sum_{j=1}^{r}\binom{r}{j}f^{(r-j)}(0)A(T_1,\ldots,T_j)\right\},\tag{10}$$

where T_m is defined in the statement of the Theorem. To complete the proof, we note that b_{n+r} is the coefficient of x in the expression

$$\frac{f(x)}{x^{r-1}(x-1)\cdots\left(x-\dfrac{q^n-1}{q-1}\right)},$$

which, from Lemma 2, is equal to

$$b_{n+r}=\sum_{j=1}^{m-n-r+1}\frac{f^{(n+j+r-1)}(0)}{(n+j+r-1)!}S_q(n+j-1,n).\tag{11}$$

Substituting (10) and (11) in (7), we get the result. □

Now we shall obtain some consequences of this result. The first of them which is stated in [1, p. 220] is

Corollary 1. *Let n be a positive integer. Then*

$$\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}q^{\frac{1}{2}i(i-1)}x^i=(1+x)(1+qx)\cdots(1+q^{n-1}x).$$

Proof. Let j, $1 \leqslant j \leqslant n$ be a fixed integer, let $f(x) = x(1+(q-1)x)^{j-1}$ and let $r = 1$. Then from Theorem 1 we get

$$\sum_{i=1}^{n} \frac{(-1)^{i+1}\begin{bmatrix} n \\ i \end{bmatrix} q^{\frac{1}{2}i(i-1)}}{q^{i(n-j)}} = 1$$

or

$$\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} q^{\frac{1}{2}i(i-1)} \left(\frac{-1}{q^{n-j}} \right)^{i} = 0, \quad \text{for} \quad j = 1, 2, \ldots, n.$$

Then $-1, -1/q, \ldots, -1/q^{n-1}$ are the zeros of the equation $\sum_{i=0}^{n} \left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right] q^{\frac{1}{2}i(i-1)} x^i = 0$, and thus we have the results. $\square$

The next results is a generalization of Theorem 1 of [2].

Corollary 2. *Let n, r be any positive integers. Let f be any polynomial of degree m. Then*

$$\sum_{i=1}^{n} \frac{(-1)^{i+1}\binom{n}{i} f(i)}{i^r}$$
$$= \frac{1}{r!} \left\{ f^{(r)}(0) + \sum_{i=1}^{r} \binom{r}{i} f^{(r-i)}(0) A(T_1^*, \ldots, T_i^*) \right\}$$
$$+ (-1)^{n+1} n! \sum_{j=1}^{m-n-r+1} \frac{f^{(n+j+r-1)}(0)}{(n+j+r-1)!} S(n+j-1, n),$$

where for $1 \leqslant m \leqslant r$, $T_m^* = \sum_{j=1}^{n} 1/j^m$ and $S(m, k)$ are Stirling numbers of the second kind.

Proof. If we let q approach 1 in the equation of the Theorem 1, we get the result. $\square$

References

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[3] J. Riordan, An Introduction to Combinatorial Analysis (Princeton University Press, Princeton, NY, 1978).